Theoretical Question 1:

"Ping-Pong" Resistor

A capacitor consists of two circular parallel plates both with radius R separated by distance d, where $d << R$, as shown in Fig. 1.1(a). The top plate is connected to a *constant* voltage source at a potential V while the bottom plate is grounded. Then a thin and small disk of mass m with radius r ($<< R, d$) and thickness t ($<< r$) is placed on the center of the bottom plate, as shown in Fig. 1.1(b).

Let us assume that the space between the plates is in vacuum with the dielectric constant ε_0; the plates and the disk are made of perfect conductors; and all the electrostatic edge effects may be neglected. The inductance of the whole circuit and the relativistic effects can be safely disregarded. The image charge effect can also be neglected.

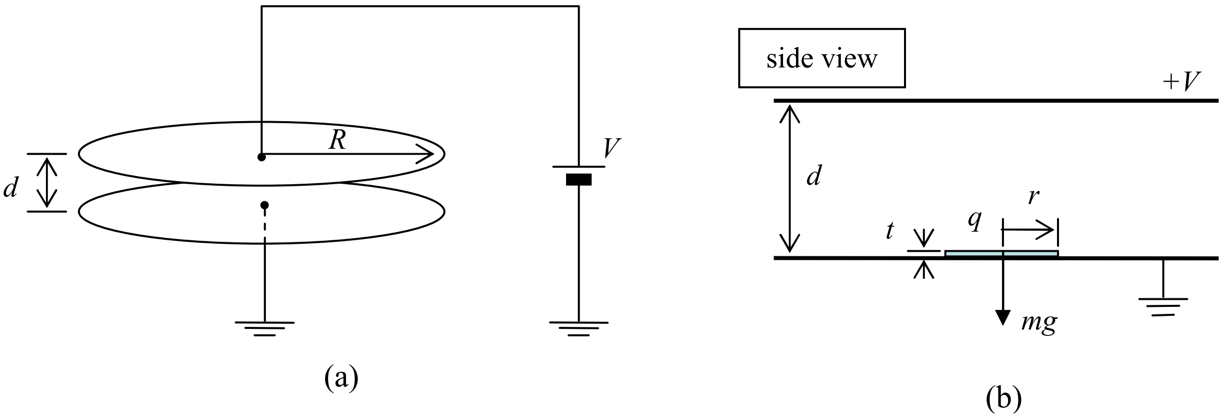

Figure 1.1 Schematic drawings of (a) a *parallel plate* capacitor connected to a constant voltage source and (b) a side view of the parallel *plates* with a small *disk* inserted inside the capacitor. (See text for details.)

(a) [1.2 points] Calculate the electrostatic force F_p between *the plates* separated by d before inserting the disk in-between as shown in Fig. 1.1(a).

(b) [0.8 points] When the disk is placed on the bottom plate, a charge q on *the disk* of Fig. 1.1(b) is related to the voltage V by $q = \chi V$. Find χ in terms of r, d, and ε_0.

(c) [0.5 points] The parallel plates lie perpendicular to a uniform gravitational field g. To lift up the disk at rest initially, we need to increase the applied voltage beyond a threshold voltage V_{th}. Obtain V_{th} in terms of m, g, d, and χ.

(d) [2.3 points] When $V > V_{\text{th}}$, the disk makes an up-and-down motion between the plates. (Assume that the disk moves only vertically *without any wobbling.*) The collisions between the disk and the plates are inelastic with the restitution coefficient $\eta \equiv (\mathrm{v}_{\text{after}} / \mathrm{v}_{\text{before}})$, where $\mathrm{v}_{\text{before}}$ and $\mathrm{v}_{\text{after}}$ are the speeds of the disk just before and after the collision respectively. The plates are stationarily fixed in position. The speed of the disk *just after* the collision at the bottom plate approaches a "steady-state speed" v_{s}, which depends on V as follows:

$$\mathrm{v}_{\text{s}} = \sqrt{\alpha V^2 + \beta}\,. \tag{1.1}$$

Obtain the coefficients α and β in terms of m, g, χ, d, and η. Assume that the whole surface of the disk touches the plate evenly and simultaneously so that the complete charge exchange happens instantaneously at every collision.

(e) [2.2 points] After reaching its steady state, the time-averaged current I through the capacitor plates can be approximated by $I = \gamma V^2$ when $qV >> mgd$. Express the coefficient γ in terms of m, χ, d, and η.

(f) [3 points] When the applied voltage V is decreased (extremely slowly), there exists a critical voltage V_{c} below which the charge will cease to flow. Find V_{c} and the corresponding current I_{c} in terms of m, g, χ, d, and η. By comparing V_{c} with the lift-up threshold V_{th} discussed in (c), make a rough sketch of the $I-V$ characteristics when V is increased and decreased in the range from $V = 0$ to $3V_{\text{th}}$.

Theoretical Question 2

Rising Balloon

A rubber balloon filled with helium gas goes up high into the sky where the pressure and temperature decrease with height. In the following questions, assume that the shape of the balloon remains spherical regardless of the payload, and neglect the payload volume. Also assume that the temperature of the helium gas inside of the balloon is always the same as that of the ambient air, and treat all gases as ideal gases. The universal gas constant is R =8.31 J/mol·K and the molar masses of helium and air are $M_H = 4.00 \times 10^{-3}$ kg/mol and $M_A = 28.9 \times 10^{-3}$ kg/mol, respectively. The gravitational acceleration is $g = 9.8$ m/s^2.

[Part A]

(a) [1.5 points] Let the pressure of the ambient air be P and the temperature be T. The pressure inside of the balloon is higher than that of outside due to the surface tension of the balloon. The balloon contains n moles of helium gas and the pressure inside is $P + \Delta P$. Find the buoyant force F_B acting on the balloon as a function of P and ΔP.

(b) [2 points] On a particular summer day in Korea, the air temperature T at the height z from the sea level was found to be $T(z) = T_0(1 - z/z_0)$ in the range of $0 < z < 15$ km with z_0 =49 km and T_0 =303 K. The pressure and density at the sea level were P_0 = 1.0 atm = 1.01×10^5 Pa and ρ_0 = 1.16 kg/m^3, respectively. For this height range, the pressure takes the form

$$P(z) = P_0(1 - z/z_0)^{\eta} \quad . \tag{2.1}$$

Express η in terms of z_0, ρ_0, P_0, and g, and find its numerical value to the *two* significant digits. Treat the gravitational acceleration as a constant, independent of height.

[Part B]

When a rubber balloon of spherical shape with un-stretched radius r_0 is inflated to a sphere of radius r ($\geq r_0$), the balloon surface contains extra elastic energy due to the stretching. In a simplistic theory, the elastic energy at constant temperature T can be expressed by

$$U = 4\pi r_0^2 \kappa RT (2\lambda^2 + \frac{1}{\lambda^4} - 3) \tag{2.2}$$

where $\lambda \equiv r/r_0$ (≥ 1) is the size-inflation ratio and κ is a constant in units of mol/m^2.

(c) [2 points] Express ΔP in terms of parameters given in Eq. (2.2), and sketch ΔP as a function of $\lambda = r/r_0$.

(d) [1.5 points] The constant κ can be determined from the amount of the gas needed to inflate the balloon. At T_0=303 K and P_0=1.0 atm = 1.01×10^5 Pa, an un-stretched balloon ($\lambda = 1$) contains n_0=12.5 moles of helium. It takes n=3.6n_0=45 moles in total to inflate the balloon to λ=1.5 at the same T_0 and P_0. Express the balloon parameter a, defined as $a = \kappa/\kappa_0$, in terms of n, n_0, and λ, where $\kappa_0 \equiv \dfrac{r_0 P_0}{4RT_0}$. Evaluate a to the two significant digits.

[Part C]

A balloon is prepared as in (d) at the sea level (inflated to $\lambda = 1.5$ with $n = 3.6n_0 = 45$ moles of helium gas at T_0=303 K and P_0=1 atm=1.01×10^5 Pa). The total mass including gas, balloon itself, and other payloads is $M_{\mathrm{T}} = 1.12$ kg. Now let the balloon rise from the sea level.

(e) [3 points] Suppose that the balloon eventually stops at the height z_f where the buoyant force balances the total weight. Find z_f and the inflation ratio λ_f at that height. Give the answers in two significant digits. Assume there are no drift effect and no gas leakage during the upward flight.

Theoretical Question 3

Atomic Probe Microscope

Atomic probe microscopes (APMs) are powerful tools in the field of nano-science. The motion of a cantilever in APM can be detected by a photo-detector monitoring the reflected laser beam, as shown in Fig. 3.1. The cantilever can move only in the vertical direction and its displacement z as a function of time t can be described by the equation

$$m\frac{d^2z}{dt^2}+b\frac{dz}{dt}+kz=F\,, \tag{3.1}$$

where m is the cantilever mass, $k=m\omega_0^2$ is the spring constant of the cantilever, b is a small damping coefficient satisfying $\omega_0 >> (b/m) > 0$, and finally F is an external driving force of the piezoelectric tube.

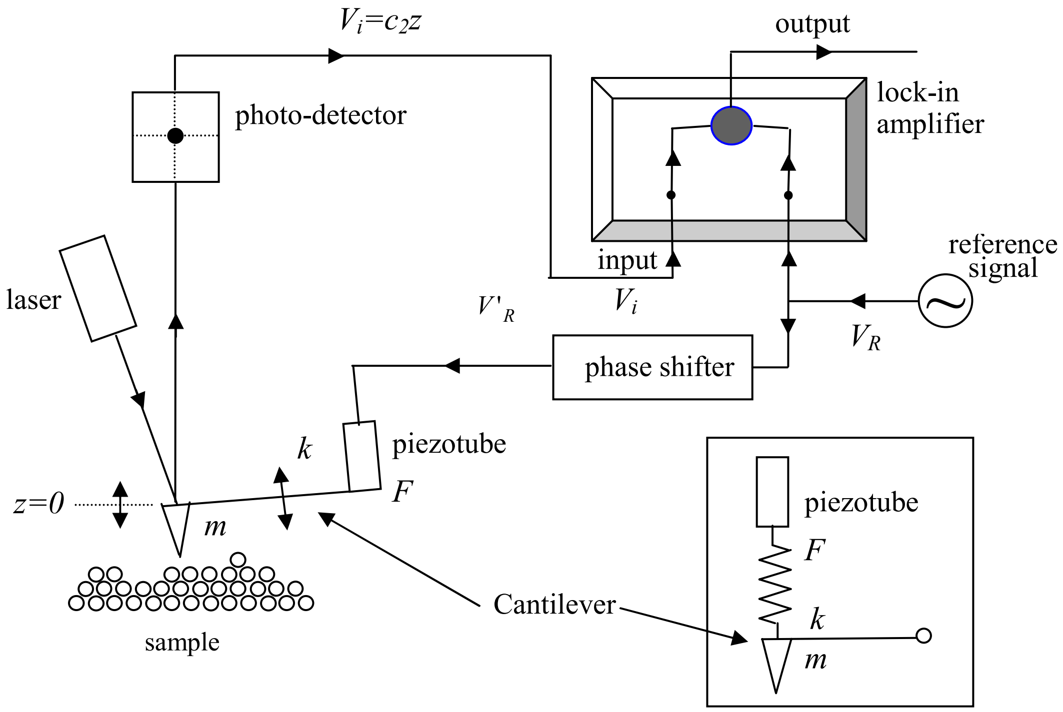

Figure 3.1 A schematic diagram for a scanning probe microscope (SPM). The inset in the lower right corner represents a simplified mechanical model to describe the coupling of the piezotube with the cantilever.

[Part A]

(a) [1.5 points] When $F=F_0\sin\omega t$, $z(t)$ satisfying Eq. (3.1) can be written as $z(t)=A\sin(\omega t-\phi)$, where $A>0$ and $0\le\phi\le\pi$. Find the expression of the amplitude A and $\tan\phi$ in terms of F_0, m, ω, ω_0, and b. Obtain A and the phase ϕ at the resonance frequency $\omega = \omega_0$.

(b) [1 point] A lock-in amplifier shown in Fig.3.1 multiplies an input signal by the lock-in reference signal, $V_R = V_{R0}\sin\omega t$, and then passes *only* the dc (direct current) component of the multiplied signal. Assume that the input signal is given by $V_i = V_{i0}\ \sin(\omega_i t - \phi_i)$. Here V_{R0}, V_{i0}, ω_i, and ϕ_i are all positive given constants. Find the condition on ω (>0) for a non-vanishing output signal. What is the expression for the magnitude of the non-vanishing *dc output signal* at this frequency?

(c) [1.5 points] Passing through the phase shifter, the lock-in reference voltage $V_R = V_{R0}\sin\omega t$ changes to $V'_R = V_{R0}\sin(\omega t + \pi/2)$. V'_R, applied to the piezoelectric tube, drives the cantilever with a force $F = c_1 V'_R$. Then, the photo-detector converts the displacement of the cantilever, z, into a voltage $V_i = c_2 z$. Here c_1 and c_2 are constants. Find the expression for the magnitude of the *dc output signal* at $\omega = \omega_0$.

(d) [2 points] The small change Δm of the cantilever mass shifts the resonance frequency by $\Delta\omega_0$. As a result, the phase ϕ at the original resonance frequency ω_0 shifts by $\Delta\phi$. Find the mass change Δm corresponding to the phase shift $\Delta\phi = \pi/1800$, which is a typical resolution in phase measurements. The physical parameters of the cantilever are given by $m = 1.0\times10^{-12}$ kg, $k = 1.0$ N/m, and $(b/m) = 1.0\times10^{3}$ s^{-1}. Use the approximations $(1+x)^a \approx 1 + ax$ and $\tan(\pi/2 + x) \approx -1/x$ when $|x| << 1$.

[Part B]

From now on let us consider the situation that some forces, besides the driving force discussed in Part A, act on the cantilever due to the sample as shown in Fig.3.1.

(e) [1.5 points] Assuming that the additional force $f(h)$ depends only on the distance h between the cantilever and the sample surface, one can find a new equilibrium position h_0. Near $h = h_0$, we can write $f(h) \approx f(h_0) + c_3(h - h_0)$, where c_3 is a constant in h. Find the new resonance frequency ω'_0 in terms of ω_0, m, and c_3.

(f) [2.5 points] While scanning the surface by moving the sample horizontally, the tip of the cantilever charged with $Q = 6e$ encounters an electron of charge $q = e$ trapped

(localized in space) at some distance below the surface. During the scanning around the electron, the maximum shift of the resonance frequency $\Delta\omega_0 (= \omega'_0 - \omega_0)$ is observed to be much smaller than ω_0. Express the distance d_0 from the cantilever to the trapped electron at the maximum shift in terms of m, q, Q, ω_0, $\Delta\omega_0$, and the Coulomb constant k_e. Evaluate d_0 in nm (1 nm = 1×10^{-9} m) for $\Delta\omega_0 = 20$ s^{-1}.
The physical parameters of the cantilever are $m = 1.0\times10^{-12}$ kg and $k = 1.0$ N/m. Disregard any polarization effect in both the cantilever tip and the surface. Note that $k_e = 1/4\pi\varepsilon_0 = 9.0\times10^9$ N·m^2/C^2 and $e = -1.6\times10^{-19}$ C.